Northern Beaches Secondary College

Manly Selective Campus

2011
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators and templates may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1- 8
- All questions are of equal value

Marks

Question 1 ***(Answer in a separate booklet)*** **15**

(a) $\int x \sec^2 (x^2)\, dx$ *(1)*

(b) $\int \frac{dx}{x(1+x^2)}$ *(3)*

(c) Use the substitution $t = \tan\frac{x}{2}$ calculate $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 3\sin x + 4\cos x}$ *(4)*

(d) $\int \frac{dx}{\sqrt{x+5}\,\sqrt{4-x}}$ for $x < 4$ using the substitution $u^2 = 4 - x$. *(3)*

(e) $\int \sin(\ln x)\, dx$ using $u = \ln x$ *(4)*

Marks

Question 2 ***(Answer in a separate booklet)*** **15**

(a) Simplify:

(i) $(4 + 3i)^2$ *(1)*

(ii) $\dfrac{7 - 2i}{3 + i}$ *(1)*

(iii) $2 \text{ cis } \dfrac{\pi}{6} \times 3 \text{ cis } \dfrac{\pi}{3}$ *(1)*

(b) Find the roots of $z^5 - i = 0$ and sketch them on an argand diagram. *(3)*

(c) (i) In the same diagram, sketch the locus of both $|z - 2| = 2$ and $|z| = |z - 4i|$. *(2)*

(ii) What is the complex number represented by the point of intersection of these two loci? *(1)*

(d) Let $y = i(1 - i\sqrt{3})(\sqrt{3} + i)$

(i) Express y in cis form. *(2)*

(ii) Hence find $y^3 + y^{-3}$ in the form A + iB. *(2)*

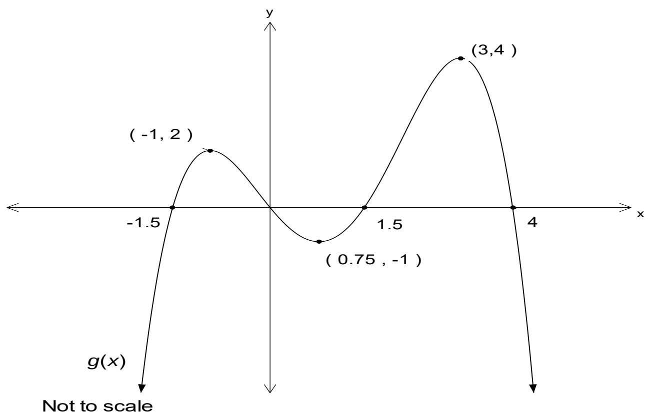
(e) Let z be a complex number of modulus 3 and ω be a complex number of modulus 1.

Show that $|z - \omega|^2 = 10 - \left(z\overline{\omega} + \bar{z}\omega\right)$ *(2)*

Marks

Question 3 ***(Answer in a separate booklet)*** **15**

(a) The diagram below shows the graph of *g(x)*.



Using this information, sketch the graphs:

(i) $k(x) = g(|x|)$ *(1)*

(ii) $t(x) = \dfrac{1}{g(x)}$ *(1)*

(iii) Graph *f(x)* given $f(x) = \left(x - \dfrac{\pi}{2}\right)\cos x$ *(3)*

(b) Given the point $P\,(6\cos\theta\,, 2\sin\theta)$ lies on an ellipse, determine the following

(i) The eccentricity. *(1)*
(ii) Coordinates of the foci. *(1)*
(iii) Equations to the directrices. *(1)*
(iv) Determine the gradient to the ellipse when $\theta = \dfrac{2\pi}{3}$. *(2)*

(c) Given the polynomial $P(x) = 2x^3 + 3x^2 - x + 1$ has roots α, β and γ:

(i) Find the polynomial whose roots are α^2, β^2 and γ^2. *(2)*

(ii) Determine the value of $\alpha^3 + \beta^3 + \gamma^3$. *(3)*

Marks

Question 4 ***(Answer in a separate booklet)*** **15**

(a) Factorise the polynomial $P(x) = 3x^3 - 7x^2 + 8x - 2 = 0$ over $\mathbb{C}$. *(2)*

(b) Using the method of taking slices parallel to the x-axis, calculate the volume of the solid of revolution when the region bounded by the curve $y = \frac{1}{2}\sqrt{x-2}$, the x-axis and the line x = 6 is rotated around the line x = 6. *(4)*

(c) The area enclosed between the curves $y = (x-4)^2$ and $y = x + 2$ is rotated about the y-axis.

(i) Draw a diagram to show the area. *(1)*

(ii) By taking slices of the area parallel to the axis of rotation, show that the volume of the solid formed is given by $2\pi \int_2^7 9x^2 - x^3 - 14x.\, dx$ *(2)*

(iii) Find the volume of the solid formed. *(1)*

(d) A solid has as its base the region bounded by the curves $y = x$ and $x = 2y - \frac{y^2}{2}$.

Cross sections parallel to the x axis are equilateral triangles with a side in the base.

Determine the volume of this solid. *(5)*

Marks

Question 5 ***(Answer in a separate booklet)*** **15**

(a) (i) If $I_n = \int_0^1 (1 - x^2)^{\frac{n}{2}} \, dx$ where n is a positive integer, show that *(4)*

$$I_n = \frac{n}{(n+1)} I_{n-2} .$$

(ii) Hence evaluate I_5. *(2)*

(b) (i) The polynomial $f(x) = x^4 - 6x^3 + 13x^2 - ax - b$ has two double zeros α and β. Find the values of a and b. *(4)*

(ii) Hence determine, with full explanation, the equation of the line which touches the curve

$$y = x^4 - 6x^3 + 13x^2$$

at two distinct points. *(1)*

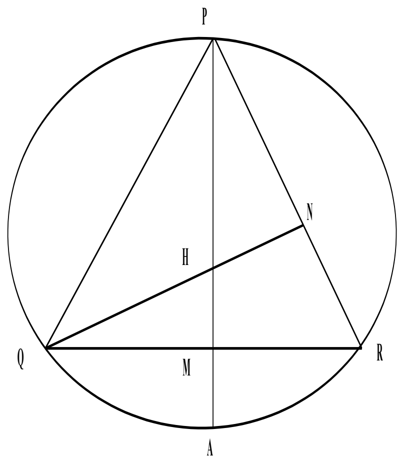
(c) If $x + \frac{1}{x} = t$, find $x^6 + \frac{1}{x^6}$ in terms of t. *(4)*

Marks

Question 6 ***(Answer in a separate booklet)*** **15**

(a) The points P, Q and R lie on the circumference of a circle and form an acute angled triangle.

The altitudes PM and QN meet at H (which is <u>not</u> the centre of the circle)



(i) Prove that $R\hat{Q}A = R\hat{Q}N$ *(3)*

(ii) Prove that HM = MA *(1)*

(iii) Prove that RH produced meets PQ at right angles. *(2)*

(b) (i) Show that $a^2 + b^2 \geq 2ab$ where a and b are distinct positive real numbers. *(1)*

(ii) Hence show that $a^2 + b^2 + c^2 \geq ab + ac + bc$. *(1)*

(iii) Hence show that $\sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$. *(2)*

(iv) Hence show that $\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha - \cos\alpha + \sec\alpha + \frac{1}{2}\sin 2\alpha$. *(2)*

(c) One of the roots of the equation $kxe^{-x} - 4 = 0$ is a double root.
Find the value of k. *(3)*

Marks

Question** 7 (Answer in a separate booklet)* ***15

(a) Determine the equation to the tangent to the curve $x^3 + y^3 - 3x^2y^2 = 1$

at the point P (1,3). *(3)*

(b) A railway line has been constructed around a circular curve of radius 800metres, and is banked by raising the outer rail to a certain level above the inner rail.

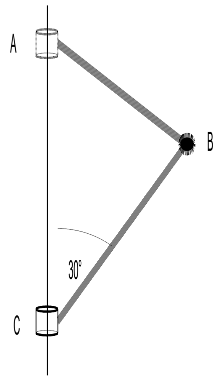
(i) When the train travels at 20m/s, the lateral thrust F, is on the outer rail. Show that

$F = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$ where Θ is the angle of inclination. *(2)*

(ii) When the train travels at 10m/s, the lateral thrust on the inner rail is the same as the lateral thrust on the outer rail at a speed of 20m/s.

a) Find the angle of the banking. *(2)*

b) Find the speed of the train when there is no lateral thrust exerted on the rails. Use $g = 9.8ms^{-2}$ *(2)*



(c) The above diagram shows a mass of 10 kilograms at B connected by light rods (at right angles) to sleeves A and C which revolve freely about the vertical axis AC but do not move vertically. The angle between the vertical axis AC and the light rod BC is 30°. The acceleration due to gravity is 9.8 m/sec^2.

(i) Given AC is 2 metres, show that the radius of the circular path of rotation of B is $\frac{\sqrt{3}}{2}$ metres. *(1)*

(ii) Find the tensions in the rods AB and BC when the mass makes 90 revolutions per minute about the vertical axis. *(5)*

Marks

Question 8 ***(Answer in a separate booklet)*** **15**

(a) Given the ellipse $\frac{x^2}{225}+\frac{y^2}{144}=1$, prove that the section of the tangent between the point of contact and its point of intersection with the directrix subtends a right angle with the corresponding focus.
(nb. The equation to the tangent to the ellipse is $\frac{x\,x_1}{a^2}+\frac{y\,y_1}{b^2}=1$ and does need to be proved.) *(3)*

(b) The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with eccentricity e, has one focus S on the positive x-axis and the corresponding directrix d cuts the asymptotes to the hyperbola at points P and Q in the first and fourth quadrants respectively.

(i) Show that PS is perpendicular to the asymptote through P and that $PS=b$. *(3)*

(ii) A circle with centre S touches the asymptotes of the hyperbola. Deduce that the points of contact are the points P and Q. *(1)*

(iii) The circle with centre S which touches the asymptotes of the hyperbola cuts the hyperbola at points R and T. If $b=a$, show that RT is a diameter. *(2)*

(c) When $(1+ax)^5+(1+bx)^5$ in expanded in ascending powers of x, the expansion begins

$$2+30x+220x^2+\ldots$$

(i) Prove that $(a+b)=6$ and $(a^2+b^2)=22$ *(2)*

(ii) Deduce the value of ab. *(2)*

(iii) Determine the value of the coefficient of x^3. *(2)*

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

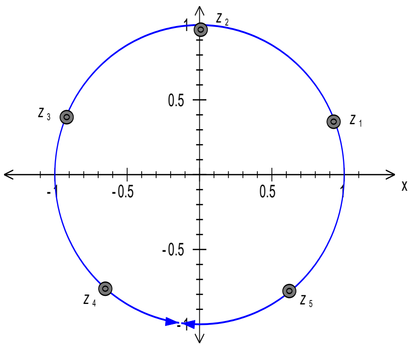
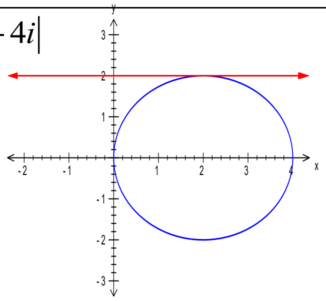
NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1:

a)	$\int x\sec^2(x^2).dx$ Let $u = x^2$ $\therefore du = 2x.dx$ $\therefore I_0 = \frac{1}{2}\int \sec^2 u.du$ $= \frac{1}{2}\tan u + c = \frac{1}{2}\tan x^2 + c$	*1 mark – correct answer*
b)	$\int \frac{dx}{x(1+x^2)}$ Let $\frac{1}{x(1+x^2)} = \frac{A}{x} + \frac{Bx+C}{1+x^2}$ $\therefore 1 = A(1+x^2) + x(Bx+C)$ Sub $x=0$: $1 = A+0 \rightarrow A=1$ Equate x^2: $0 = A+B \rightarrow B=-1$ Equate x: $0 = C \rightarrow C=0$ $\therefore \int \frac{dx}{x(1+x^2)} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right).dx$ $= \ln x - \frac{1}{2}\ln(x^2+1) + c$	*1 mark for setting up partial fractions* *1 mark for finding constants* *1 mark for answer* *Comment: Many students gained the first 2 marks but left the x out of the final integral*
c)	$\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\sin x+4\cos x}$ Let $t = \tan\frac{x}{2}$ At $x=0$, $t=0$ $\therefore x = 2\tan^{-1}t$, $x=\frac{\pi}{2}$, $t=1$ and $dx = \frac{2dt}{1+t^2}$, $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ $\therefore I = \int_0^1 \frac{\frac{2dt}{1+t^2}}{5+3\left(\frac{2t}{1+t^2}\right)+4\left(\frac{1-t^2}{1+t^2}\right)}$ $= \int_0^1 \frac{2dt}{5+5t^2+6t+4-4t^2}$ $= \int_0^1 \frac{2dt}{(t+3)^2}$ $= -2\left[\frac{1}{t+3}\right]_0^1 = 2\left[\frac{1}{t+3}\right]_1^0 = 2\left(\frac{1}{3}-\frac{1}{4}\right) = \frac{1}{6}$	*1 mark – sub in correct expressions* *1 mark – correct integral* *1 mark – integration* *1 mark - substitution* *Comment: many students lost the 2 next to the dt*

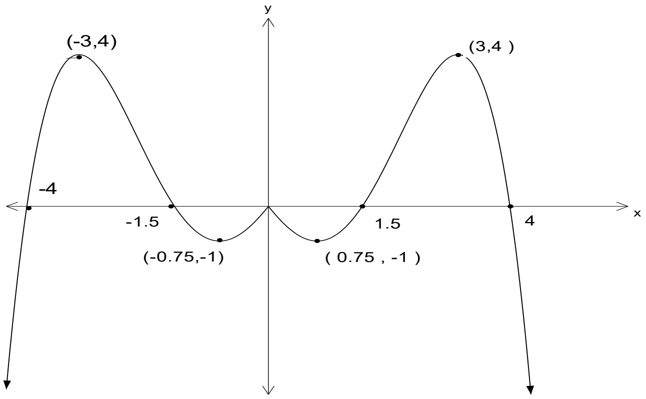
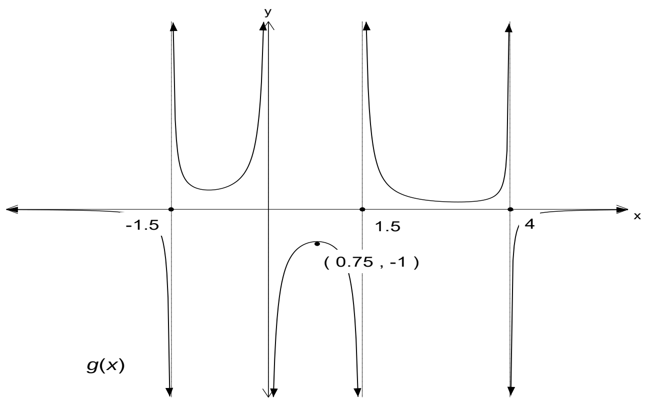
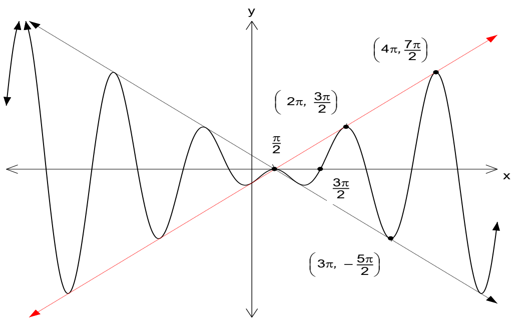
d)	$\int \frac{dx}{\sqrt{x+5}.\sqrt{4-x}}$ Let $u^2 = 4 - x \rightarrow 2u.du = -dx$ and $x = 4 - u^2$ $\therefore I = \int \frac{-2du}{\sqrt{9-u^2}.\sqrt{u^2}} = -\int \frac{2du}{\sqrt{9-u^2}}$ $= -2\sin^{-1}\left(\frac{u}{3}\right) + c = -2\sin^{-1}\left(\frac{\sqrt{4-x}}{3}\right) + c$ Or $= 2\cos^{-1}\left(\frac{u}{3}\right) + c = 2\cos^{-1}\left(\frac{\sqrt{4-x}}{3}\right) + c$	*1 mark – correct substitution* *1 mark – set up correct integral* *1 mark – correct answer* *Comment: there was mixing up of u's and x's*
e)	$\int \sin(\ln x)\, dx$ Let $u = \ln x \rightarrow x = e^u$ and $dx = e^u du$ $\therefore I = \int \sin(u).e^u.du$ Let $U = e^u$, $dV = \sin u.du$ $\therefore dU = e^u du$, $V = -\cos u$ $\therefore \int \sin(u).e^u.du = -e^u \cos u + \int e^u \cos u.du$ Let $U = e^u$, $dV = \cos u.du$ $\therefore dU = e^u du$, $V = \sin u$ $\therefore \int \sin(u).e^u.du = -e^u \cos u + e^u \sin u - \int e^u \sin u.du$ $\therefore 2\int e^u \sin u.du = -e^u \cos u + e^u \sin u$ $\therefore \int e^u \sin u.du = \frac{1}{2}\left(e^u \sin u - e^u \cos u\right) + c$ i.e $\int \sin(\ln x)\, dx = \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + c$	*1 mark for correct substitution* *1 mark for correct use in integration by parts* *1 mark for Int by Parts correct answer* *1 marks for final answer* *Comment: Setting out was often not clear*

Question 2

2ai)	$(4+3i)^2 = 16+24i-9 = 7+24i$	1 mark for correct answer
ii)	$\dfrac{7-2i}{3+i} = \dfrac{7-2i}{3+i} \times \dfrac{3-i}{3-i} = \dfrac{21-6i-7i-2}{9+1} = \dfrac{19-13i}{10}$	1 mark for correct answer
iii)	$2cis\dfrac{\pi}{3} \times 3cis\dfrac{\pi}{6} = 6cis\dfrac{\pi}{2} = 6i$	1 mark for correct answer
b)	$z^5 - i = 0 \rightarrow z^5 = i$ Let $z = \cos\theta + i\sin\theta$ $\therefore z^5 = \cos 5\theta + i\sin 5\theta = i$ Equate real and imaginary parts: $\cos 5\theta = 0$ and $\sin 5\theta = 1$ $\therefore 5\theta = \dfrac{\pi}{2}, \dfrac{5\pi}{2}, \dfrac{9\pi}{2}, \dfrac{13\pi}{2}, \dfrac{17\pi}{2}$ and $\theta = \dfrac{\pi}{10}, \dfrac{\pi}{2}, \dfrac{9\pi}{10}, \dfrac{13\pi}{10}, \dfrac{17\pi}{10}$ $\therefore z_1 = cis\dfrac{\pi}{10}, z_2 = i, z_3 = cis\dfrac{9\pi}{10}, z_4 = cis\dfrac{13\pi}{10}, z_5 = cis\dfrac{17\pi}{10}$ 	1 mark for 1 correct value of z 1 mark for 5 correct values of z 1 mark for plotting the values, showing equal moduli and equal imaginary parts for z_1, z_3 and z_4, z_5 Comment: This question was marked very generously
ci)	$\|z-2\| = 2, \|z\| = \|z-4i\|$ 	1 mark for each graph
ii)	The line and curve intersect at $2+2i$	1 mark for correct answer
di)	$y = i\left(1-i\sqrt{3}\right)\left(\sqrt{3}+i\right)$ $= \left(i+\sqrt{3}\right)\left(\sqrt{3}+i\right) = 2+2i\sqrt{3} = 4cis\dfrac{\pi}{3}$	1 mark for correct expansion 1 mark for correct answer

ii)	$y^3 + y^{-3} = \left(4cis\frac{\pi}{3}\right)^3 + \left(4cis\frac{\pi}{3}\right)^{-3}$ $= 64cis\pi + \frac{1}{64}cis(-\pi)$ $= -(64 + \frac{1}{64})$	1 marks for correct use of De Moivre's theorem 1 mark for correct answer
e)	Since $z\overline{z} = \lvert z\rvert^2$, $\lvert z-w\rvert^2 = (z-w)\overline{(z-w)}$ $= (z-w)(\overline{z}-\overline{w})$ $= z\overline{z} - z\overline{w} - w\overline{z} + w\overline{w}$ $= 9 - (z\overline{w} + w\overline{z}) + 1 = 10 - (z\overline{w} + w\overline{z})$	1 mark for knowledge of conjugates 1 mark for correct answer Comment: There was only one correct response

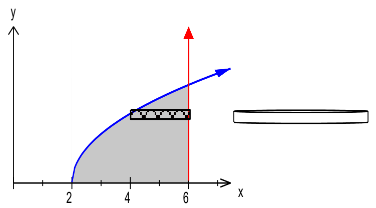
Question 3:

(a)i		*1 mark – correct answer clearly showing symmetry.*
(a)ii	Not to scale  Not to scale	*1 mark – correct answer including showing asymptotes*
(a)iii		*3 marks – correct answer clearly showing shape (eg. Domain approx.* $-4\pi \le x \le 4\pi$ *or greater) and upper and lower asymptotes or some indication of max/min values* *2 marks –clearly showing shape (eg. Domain approx.* $-4\pi \le x \le 4\pi$ *or greater)* *1mark – significant process towards demonstrating shape.*

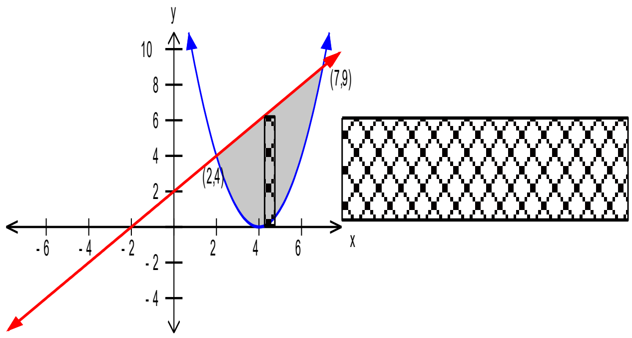
b-(i)	$P(6\cos\theta, 2\sin\theta)\ \therefore \frac{x^2}{36}+\frac{y^2}{4}=1$ $b^2 = a^2(1-e^2)$ $e=\sqrt{1-\frac{4}{36}}$ $=\sqrt{\frac{32}{36}}=\frac{2\sqrt{2}}{3}$	*1 mark – correct answer*
b-(ii)	$Foci = (\pm ae, 0) = (\pm 4\sqrt{2}, 0)$	*1 mark – correct answer*
b-(iii)	$Directrices\ x = \pm\frac{a}{e}$ $x=\pm\frac{6\times 3}{2\sqrt{2}}=\pm\frac{9\sqrt{2}}{2}$	*1 mark – correct answer*
b-(iv)	$\frac{x^2}{36}+\frac{y^2}{4}=1$ $\frac{2x}{36}+\frac{2y}{4}\frac{dy}{dx}=0$ $\quad x=6\cos\theta\ y=2\sin\theta$ $\frac{dy}{dx}=-\frac{x}{9y}$ $\quad \frac{dx}{d\theta}=-6\sin\theta$ $\quad \frac{dy}{d\theta}=2\cos\theta$ $\theta=\frac{2\pi}{3}$ $\quad \frac{dy}{dx}=-\frac{1}{3}\cot\theta$ $x=-3\quad y=\sqrt{3}$ $\quad \frac{dy}{dx}=\frac{1}{3\sqrt{3}}=\frac{\sqrt{3}}{9}$ $m=\frac{1}{3\sqrt{3}}=\frac{\sqrt{3}}{9}$	*2 mark – correct answer* *1 mark – correct equation for* $\frac{dy}{dx}$
(c) Ii	$P(x)=2x^3+3x^2-x+1$ $Q(x)=2(\sqrt{x})^3+3(\sqrt{x})^2-\sqrt{x}+1$ $0=2(\sqrt{x})^3+3(\sqrt{x})^2-\sqrt{x}+1$ $2(\sqrt{x})^3-\sqrt{x}=-3x-1$ $\sqrt{x}(2x-1)=-(3x+1)$ $x(2x-1)^2=(3x+1)^2$ $4x^3-4x^2+x=9x^2+6x+1$ $Q(x)=4x^3-13x^2-5x-1$	***2 marks – correct answer*** ***1 mark – correct substitution of*** $\sqrt{x}$ **.**

(d) Iii	$P(\alpha) = 2\alpha^3 + 3\alpha^2 - \alpha + 1 = 0$ $P(\beta) = 2\beta^3 + 3\beta^2 - \beta + 1 = 0$ $P(\gamma) = 2\gamma^3 + 3\gamma^2 - \gamma + 1 = 0$ $2(\alpha^3 + \beta^3 + \gamma^3) = -3(\alpha^2 + \beta^2 + \gamma^2) + (\alpha + \beta + \gamma) - 3$ $\alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a} = \frac{13}{4}$ *from part (i)* $\alpha + \beta + \gamma = -\frac{3}{2}$ $2(\alpha^3 + \beta^3 + \gamma^3) = -3 \times \frac{13}{4} - \frac{3}{2} - 3$ $\alpha^3 + \beta^3 + \gamma^3 = -\frac{57}{8}$	

Question 4

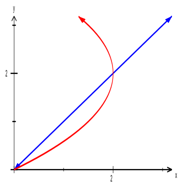
(a)	$P(x) = 3x^3 - 7x^2 + 8x - 2 = 0$ $Test \pm 1, 2, \frac{1}{3}, \frac{2}{3}$ $P\left(\frac{1}{3}\right) = 0 \therefore (3x-1)$ *is a factor* $P(x) = (3x-1)(x^2 - 2x + 2)$ *by inspection &/or long division.* $= (3x-1)[(x-1)^2 + 1]$ $= (3x-1)(x-1-i)(x-1+i)$	*2 marks – correct answer factorised over C.* *1 mark – correctly factorised over R.*
(b) (i)	 **Volume of single cylinder** $V = \pi r^2.h \qquad r = 6 - x \qquad h = \delta y$ $\delta V = \pi(6-x)^2.\delta y$ $y = \frac{1}{2}\sqrt{x-2}$ $x = 4y^2 + 2$ $\delta V = \pi(4 - 4y^2)^2.\delta y$ $= 16\pi(1 - y^2)^2.\delta y$ **Approximate volume of shape.** $V \cong \lim_{dy \to 0} \sum_{0}^{1} 16\pi(1-y^2)^2.\delta y$ $V = 16\pi \int_0^1 1 - 2y^2 + y^4.dy$ $= 16\pi\left[y - \frac{2y^3}{3} + \frac{y^5}{5}\right]_0^1$ $= \frac{128}{12}\pi\ units^3$	*4 marks – correct answer fully explained.* *3 marks – Correct integration for correct volume expression* *2 mark – approximate volume showing Σ notation* *(nb Σ notation was specifically required to be demonstrated in this solution.)* *1 mark – formula for volume of single cylinder ie. δV*

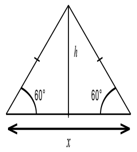
Question 4 (continued)

c-(i)		*1 mark – correct diagram including intercepts.*
c-(ii)	$\delta V = 2\pi r.\ h.\ \delta x \qquad r = x \qquad h = (x+2) - (x-4)^2$ $\therefore\ \delta V = 2\pi .\ x.\ [(x+2) - (x-4)^2].\ \delta x$ $= 2\pi x(9x - x^2 - 14).\ \delta x$ $= 2\pi(9x^2 - x^3 - 14x).\ \delta x$ Intercepts $x + 2 = (x-4)^2$ $0 = x^2 - 9x + 14$ $= (x-2)(x-7)$	*2 marks – correctly demonstrated derivation of formula* *1 mark – correct process with single error.*
c-(iii)	$V \cong \lim_{x \to 0} \sum_{2}^{7} 2\pi(9x^2 - x^3 - 14x).\ \delta x$ $V = 2\pi \int_2^7 (9x^2 - x^3 - 14x).\ dx$ $= 2\pi \left[3x^3 - \frac{x^4}{4} - 7x^2 \right]_2^7$ $= \frac{375\pi}{2}\ units^3$	*1 mark – correct answer*

Question 4 (continued)

(d)





$h = \frac{\sqrt{3}}{2}x$

Intercepts

$$y = 2y - \frac{y^2}{2}$$

$$y^2 - 2y = 0$$

$$y = 0 \qquad y = 2$$

Length of side $= x_2 - x_1$

$$= 2y - \frac{y^2}{2} - y = y - \frac{y^2}{2}$$

Area of Triangle $= \frac{1}{2} ab\sin 60^\circ$

$$= \frac{1}{2}\left(y - \frac{y^2}{2}\right)^2 . \frac{\sqrt{3}}{2}$$

Volume of single triangular prism

$$V = \frac{\sqrt{3}}{4} . \left(y - \frac{y^2}{2}\right)^2 . \delta y$$

Volume of shape.

$$V \approx \lim_{\delta y \to \infty} \sum_{y=0}^{2} \frac{\sqrt{3}}{4}\left(y - \frac{y^2}{2}\right)^2 \delta y$$

$$= \frac{\sqrt{3}}{4}\int_0^2 y^2 - y^3 + \frac{y^4}{4}\, dy$$

$$= \frac{\sqrt{3}}{4}\left[\frac{y^3}{3} - \frac{y^4}{4} + \frac{y^5}{20}\right]_0^2$$

$$= \frac{\sqrt{3}}{15} u^3$$

5 marks – correct answer clearly demonstrated.

4 marks – Correct to final integration

3 marks – correct derivation of δV

2 marks – correct formula in terms of y for Area of triangle.

1 mark – length of side in terms of y.

Question 5:

(a) (i)	$I_n = \int_0^1 (1-x^2)^{\frac{n}{2}}\,dx$ $= \left[x(1-x^2)^{\frac{n}{2}}\right]_0^1 + n\int_0^1 x^2(1-x^2)^{\frac{n}{2}-1}$ $= 0 - n\int_0^1 (1-x^2+1)(1-x^2)^{\frac{n}{2}-1}$ $= -n\int_0^1 (1-x^2)^{\frac{n}{2}}dx + n\int_0^1 \left(\frac{1}{x^2}\right)^{\frac{n}{2}-1}dx$ $= -nI_n + nI_{n-2}$ $= \frac{nI_{n-2}}{n+1}$	*4 marks – correct proof* *3 marks – error in manipulation of separate parts* *2 marks – incorporation of transformation in step 2* *1 mark – correct integration by parts*
(ii)	$I_5 = \frac{5}{6}I_3$ $I_3 = \frac{3}{4}I_1$ $I_1 = \int_0^1 \sqrt{1-x^2}\;dx$ $= \frac{\pi}{4}$ $\therefore I_3 - \frac{3}{4} \times \frac{\pi}{4}$ $I_5 = \frac{5}{6} \times \frac{3}{4} \times \frac{\pi}{4} = \frac{5\pi}{32}$	*2 marks – correct value determined* *1 mark – correct process but error*
(b) (i)	$f(x) = x^4 - 6x^3 + 13x^2 - ax - b$ $f'(x) = 4x^3 - 18x^2 + 26x - a$ Roots are $\alpha\,\alpha\,\beta\,\beta$ $\therefore 2(\alpha + \beta) = 6$ so $\alpha + \beta = 3$ $\alpha^2 + \beta^2 + 4\alpha\beta = 13$ $(\alpha + \beta)^2 + 2\alpha\beta = 13$ $9 + 2\alpha(3 - \alpha) = 13$ $\alpha^2 - 3\alpha + 2 = 0$ $\alpha = 1$ or 2 so $b = 2$ or 1 So $2(\alpha\alpha\beta + \alpha\beta\beta) = -a$ $a = 12$ $b = -4$	*4 marks – both values correct* *3 marks – correct values for α and β* *2 marks – both equations with α and β correct* *1 mark – one equation with α and β correct*
(ii)	Equation is $y = 12x - 4$ as solving the equations together gives the equation in (i) and the double zero gives the two points of contact.	*1 mark – correct integration by parts*

***Question 5** (continued)*

(c)	$\left(x+\frac{1}{x}\right)^2 = x^6 + \frac{6x^5}{x} + \frac{15x^4}{x^2} + \frac{20x^3}{x^3} + \frac{15x^2}{x^4} + \frac{6x}{x^5} + \frac{1}{x^6}$ $= \left(x^6 + \frac{1}{x^6}\right) + 6\left(x^4+\frac{1}{x^4}\right) + 15\left(x^2 + \frac{1}{x^2}\right) + 20$ $x^2 + \frac{1}{x^2} = \left(x+\frac{1}{x}\right)^2 - 2 = t^2 - 2$ $\left(x^4+\frac{1}{x^4}\right) = \left(x^2 + \frac{1}{x^2}\right)^2 - 2 = (t^2-2)^2 - 2$ $= t^4 - 4t^2 + 2$ $t^6 = \left(x^6 + \frac{1}{x^6}\right) + 6(t^4-4t^2+2) + 15(t^2-2) + 20$ $= t^6 - 6t^4 + 9t^2 - 2$	*4 marks – correct expression for t^6* *3 marks – correct substitution for t^6 (2nd last line)* *2 marks – correct expression for $x^4 + \frac{1}{x^4}$* *1 mark – correct expression for $x^2 + \frac{1}{x^2}$*

Question 6

(a)(i)	Join QA and AR $P\hat{R}Q = P\hat{A}Q = \alpha$ (angle at circumference subtended by PQ) $A\hat{Q}R = A\hat{P}R = \beta$ (as above subtended by AR) But in $\Delta MPR\ \ \alpha + \beta = 90^\circ$ $\therefore$ in $\Delta RQN\ R\hat{Q}N + N\hat{R}Q = 90^\circ$ $\therefore R\hat{Q}N = 90^\circ - \alpha = \beta = A\hat{Q}R$ $\therefore R\hat{Q}A = R\hat{Q}N$	*3 marks – correct proof with full reasoning* *2 marks – correct approach continued for a second angle* *1 mark – approach using angles at circumference properly demonstrated*
(ii)	*In* ΔMQH and ΔMQA QM is common $Q\hat{M}H = Q\hat{M}A = 90^\circ$ (*given*) $R\hat{Q}A = R\hat{Q}N$ (proved above) $\therefore \Delta MQH \equiv \Delta MQA$ (*ASA*) $\therefore$ AM = MA (opposite corresponding angles)	*1 mark – correct proof of congruence*
(iii)	*In* ΔMHR and MAR MH = MA (proved above) MR is common $H\hat{M}R = A\hat{M}R = 90^\circ$ (given) $\therefore \Delta MHR \equiv \Delta MAR$ $\therefore M\hat{A}R = M\hat{H}R$ But $A\hat{R}Q = A\hat{P}Q$ (angles at circumference subtended by AQ) RH produced meets QP at X $M\hat{H}R = X\hat{H}P$ (vertically opposite angles) $\therefore X\hat{P}Q + X\hat{H}P = 90^\circ$ $\therefore P\hat{X}H = 90^\circ$ $\therefore$ RH produced meets PQ at right angles	*2 marks – correct proof with reasons clearly stated* *1 mark – suitable approach*

Question 6 *(continued)*

(b) (i)	$(a-b)^2 \geq 0$ $a^2 + b^2 > + 2ab$	*1 mark – correct proof of inequality*
(ii)	$a^2 + b^2 > + 2ab$ $a^2 + c^2 > + 2ac$ $b^2 + c^2 > + 2bc$ $2(a^2 + b^2 + c^2) \geq 2(ab + ac + bc)$ $a^2 + b^2 + c^2 \geq ab + ac + bc$	*1 mark – correct proof of inequality*
(iii)	Let $a = \sin\alpha$ and $b = \cos\alpha$ $\therefore \sin^2\alpha + \cos^2\alpha \geq 2\sin\alpha\cos\alpha$ (using (i) $\therefore \sin^2\alpha + \cos^2\alpha \geq \sin 2\alpha$	*1 mark – correct proof of congruence*
(iv)	$\sin^2\alpha + \cos^2\alpha + \tan^2\alpha \geq \sin\alpha\cos\alpha + \sin\alpha\tan\alpha + \cos\alpha\tan\alpha$ $\text{RHS} = \frac{1}{2}\sin 2\alpha + \frac{\sin^2\alpha}{\cos\alpha} + \sin\alpha$ $= \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{1-\cos^2\alpha}{\cos\alpha}$ $= \frac{1}{2}\sin 2\alpha + \sin\alpha + \frac{1}{\cos\alpha} - \frac{\cos^2\alpha}{\cos\alpha}$ $= \frac{1}{2}\sin 2\alpha + \sin\alpha + \sec\alpha - \cos\alpha$	*2 marks – correct proof with full logic shown* *1 mark – correct approach to simplification of RHS*
(c)	$kxe^{-x} - 4 = 0$ $\therefore ke^{-x} \times 1 - kxe^{-x} = 0$ $ke^{-x} 1 - x) = 0$ $e^{-x} \neq 0 \therefore x = 1$ $\therefore \text{k} \times 1 \times \text{e}^{-1} = 4$ $k = 4e$	*3 marks - correct value for k* *2 marks – correct value for x* *1 mark – correct differentiation*

Question 7:

(a)

$$x^3 + y^3 - 3x^2y^2 = 1$$

$$3x^2 + 3y^2\frac{dy}{dx} - 3\left(2xy^2 + 2x^2y\frac{dy}{dx}\right) = 0$$

$$(3x^2 - 6xy^2) + (3y^2 - 6x^2y)\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{3(2xy^2 - x^2)}{3(y^2 - 2x^2y)}$$

at x= 1 , y=3 $\quad \frac{dy}{dx} = \frac{18-1}{9-6} = \frac{17}{3}$

$$(y - y_1) = m(x - x_1)$$

$$y - 3 = \frac{17}{3}(x - 1)$$

$$17x - 3y - 8 = 0$$

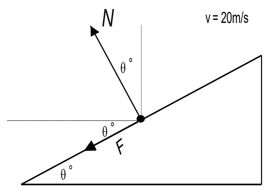
3 marks – correct answer

2 marks – correct expression for $\frac{dy}{dx}$ (gradient function).

1 mark – correct differentiation

(nb – equation formed from incorrect implicit differentiation not considered.)

(b)



$$N\cos\theta - F\sin\theta = \boldsymbol{mg} \qquad ①$$

$$N\sin\theta + F\cos\theta = \frac{mv^2}{r} \qquad ②$$

$① \times \sin\theta \quad \& \quad ② \times \cos\theta$

$$N\cos\theta\sin\theta - F\sin^2\theta = \boldsymbol{mg}\sin\theta \qquad ①$$

$$N\cos\theta\sin\theta + F\cos^2\theta = \frac{mv^2}{r}\cos\theta \qquad ②$$

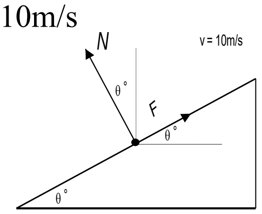
$$F(\cos^2\theta + \sin^2\theta) = \frac{mv^2}{r}\cos\theta - \boldsymbol{mg}\sin\theta$$

$$F = m\left(\frac{400}{800}\cos\theta - g\sin\theta\right)$$

$$F = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$$

2 marks – correct answer fully demonstrated.

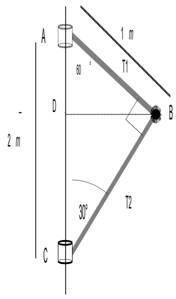
1 mark – correct resolution of forces horizontally and vertically.

<table>
<tr><td>b-(ii)</td><td>Train travelling at 10m/s


$N\cos\theta + F\sin\theta = \boldsymbol{mg}$ ①
$N\sin\theta - F\cos\theta = \frac{mv^2}{r}$ ②
① × sinθ & ② × cosθ
$N\cos\theta\sin\theta + F\sin^2\theta = \boldsymbol{mg}\sin\theta$ ③
$N\cos\theta\sin\theta - F\cos^2\theta = \frac{mv^2}{r}\cos\theta$ ④
$F(\cos^2\theta + \sin^2\theta) = \boldsymbol{mg}\sin\theta - \frac{mv^2}{r}\cos\theta$
$F = m\left(g\sin\theta - \frac{100}{800}\cos\theta\right)$
$F = m\left(g\sin\theta - \frac{1}{8}\cos\theta\right)$
$m\left(g\sin\theta - \frac{1}{8}\cos\theta\right) = m\left(\frac{1}{2}\cos\theta - g\sin\theta\right)$
$\frac{5}{8}\cos\theta = 2g\sin\theta$
$\tan\theta = \frac{5}{16g}$
$\theta = 1^\circ 50'$</td><td>2 marks – correct answer fully demonstrated.

1 mark – correct equation for F for 10m/s</td></tr>
<tr><td>v</td><td>$N\cos\theta = \boldsymbol{mg}$
$N\sin\theta = \frac{mv^2}{r}$
$\tan\theta = \frac{v^2}{gr}$
$v = \sqrt{gr\tan\theta}$
$= \sqrt{g \times 800 \times \frac{5}{16g}}$
$= 5\sqrt{10}$ m/s</td><td>2 marks – correct answer fully demonstrated.

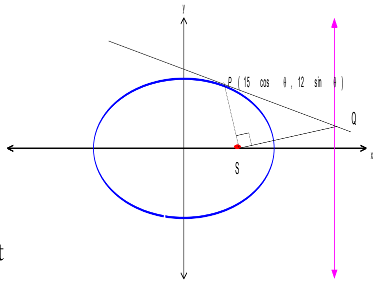
1 mark – correct derivation of tanθ</td></tr>
</table>

vi	 Given triangle ABC is 60/30 triangle then side AB = 1 Triangle ABD is also 60/30 therefore side DB = $\frac{\sqrt{3}}{2}$	1 mark – correct demonstration of length of DB
vii	$f = 90$ *rev/min* $\qquad f = \frac{\omega}{2\pi}$ *rev/sec* $\omega = \frac{180\pi}{60} = 3\pi\ s^{-1}$ Resolving forces vertically. $T_1 \cos 60 - T_2 \sin 60 = \boldsymbol{mg} = 10g$ $\frac{T_1}{2} - \frac{\sqrt{3}\,T_2}{2} = 98 \qquad ①$ Resolving forces horizontally $T_1 \sin 60 + T_2 \cos 60 = mr\omega^2$ $\frac{\sqrt{3}\,T_1}{2} + \frac{T_2}{2} = 10 \times \frac{\sqrt{3}}{2} \times (3\pi)^2 = 45\sqrt{3}\,\pi^2$ $\frac{\sqrt{3}\,T_1}{2} + \frac{T_2}{2} = 45\sqrt{3}\,\pi^2 \qquad ②$ ② $\times \sqrt{3}$ $\frac{3T_1}{2} + \frac{\sqrt{3}\,T_2}{2} = 135\pi^2$	*5 marks – correct answer showing all steps involved.* *4 marks – one correct tension determined.* *3 marks – forces resolved correctly* *2 marks – forces resolved correctly – either horizontally or vertically* *1 mark – determination for ω – angular velocity*

	$\frac{T_1}{2} - \frac{\sqrt{3}\,T_2}{2} = 98$ $\frac{3T_1}{2} + \frac{\sqrt{3}\,T_2}{2} = 135\pi^2$ $2T_1 = 98 + 135\pi^2$ $T_1 = \frac{98 + 135\pi^2}{2} = 715.2N$ $T_1 - \sqrt{3}\,T_2 = 196$ $\sqrt{3}\,T_2 = \frac{98 + 135\pi^2}{2} - 196$ $T_2 = \frac{98 + 135\pi^2 - 392}{2\sqrt{3}} = 299.76N$	

Question 8:

(a) (i)



Equation to tangent

$$\frac{x\,x_1}{a^2} + \frac{y\,y_1}{b^2} = 1 \qquad or \qquad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

$$b^2 = a^2(1 - e^2)$$

$$e^2 = 1 - \frac{144}{225}$$

$$e = \frac{3}{5}$$

$$Focus = ae = 15 \times \frac{3}{5} = 9$$

$$Directrix\ x = \frac{a}{e} = \frac{15 \times 5}{3} = 25$$

Point of intersection of tangent and directrix.

$$\frac{x\cos\theta}{15} + \frac{y\sin\theta}{12} = 1 \qquad\qquad x = 25$$

$$\frac{25\cos\theta}{15} + \frac{y\sin\theta}{12} = 1$$

$$y = \frac{12 - 20\cos\theta}{\sin\theta}$$

Gradient – Focus to P

$$m_1 = \frac{12\sin\theta}{15\cos\theta - 9} = \frac{4\sin\theta}{5\cos\theta - 3}$$

Gradient – Focus to Directrix intercept

$$m_2 = \frac{\frac{12 - 20\cos\theta}{\sin\theta}}{25 - 9} = \frac{12 - 20\cos\theta}{16\sin\theta} = \frac{3 - 5\cos\theta}{4\sin\theta}$$

$$m_1 \times m_2 = \frac{4\sin\theta}{5\cos\theta - 3} \times \frac{3 - 5\cos\theta}{4\sin\theta} = -1$$

Therefore right angle subtended at focus.

3 marks – correct answer fully demonstrated

2 marks – correct gradient for either PS or SQ

1 mark – focus and directrix correctly determined.

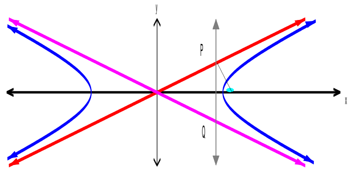
(ii)

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Focus $(ae, 0)$

Directrix $x = \frac{a}{e}$

Asymptotes $y = \pm\frac{b}{a}x$



Point P

$y = \frac{b}{a}.\frac{a}{e} = \frac{b}{e}$ therefore $P\left(\frac{a}{e}, \frac{b}{e}\right)$

Gradient PS $m = \dfrac{\frac{b}{e}}{\frac{a}{e} - ae}$

$$= \dfrac{\frac{b}{e}}{\frac{a - ae^2}{e}}$$

$$= \frac{b}{a(1 - e^2)}$$

$$= \frac{b}{a} \times -\frac{a^2}{b^2}$$

$$= -\frac{a}{b}$$

$$m_1 \times m_2 = \frac{b}{a} \times \frac{-a}{b} = -1$$

Therefore PS is perpendicular to asymptote.

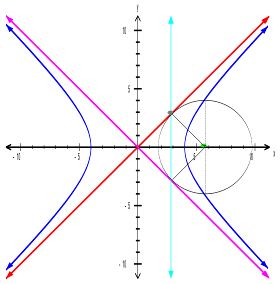
Demonstrating perpendicular.

2 marks – correct answer fully demonstrated

1 mark – correct approach and significant progress made towards reaching correct answer.

Determining Length

1 mark – correct answer determined by correct method.

	Length PS $PS = \sqrt{\left(\frac{a}{e} - ae\right)^2 + \frac{b^2}{e^2}}$ $= \sqrt{\left(\frac{a(e^2-1)}{e}\right)^2 + \frac{b^2}{e^2}}$ $= \sqrt{\frac{a^2(e^2-1)(e^2-1) + b^2}{e^2}}$ $= \sqrt{\frac{b^2(e^2-1+1)}{e^2}}$ $= b$	
(iii)	As PS and PQ are perpendicular to asymptotes and equal in length. Also radii are perpendicular to tangents at point of contact. Therefore circle with centre at S would have points of contact with aysmptotes at Points P and Q.	
(b)		*2marks – correct answer fully demonstrated* *1 mark – correct approach and significant progress made towards reaching correct answer.*

	Focus $(ae, 0)$ $\quad a = b$ *Circle with centre at focus and touching at P and Q* $(x - ae)^2 + y^2 = b^2 = a^2$ $y^2 = a^2 - (x - ae)^2$ *Hyperbola with a=b* $x^2 - y^2 = a^2$ $\therefore \quad y^2 = x^2 - a^2$ $\therefore \quad x^2 - a^2 = a^2 - (x - ae)^2$ $x^2 - a^2 = a^2 - x^2 + 2aex - a^2e^2$ $x^2 - a^2 = a^2(1 - e^2) - x^2 + 2aex$ $x^2 - a^2 = -b^2 - x^2 + 2aex$ $x^2 - a^2 = -a^2 - x^2 + 2aex \quad$ *as* $b = a$ $2x^2 - 2aex = 0$ $x = ae \quad$ *as* $x \neq 0$ Therefore x coordinate of *R, T* and *S* is the same for each point ie. $x = ae$. *Therefore are collinear* and single line through centre meets circle and hyperbola at same points, therefore RT must be a diameter.	
	Using binomial expansion gives to determine coefficient of x and x^2 ${}^5\mathbf{C}_1\, ax + {}^5\mathbf{C}_1\, bx = {}^5\mathbf{C}_1(a + b)x = 30x$ $\therefore \quad 5(a + b) = 30$ $a + b = 6$ ${}^5\mathbf{C}_2\, a^2x^2 + {}^5\mathbf{C}_2\, a^2x^2 = {}^5\mathbf{C}_2(a^2 + b^2)x^2 = 220x^2$ $10(a^2 + b^2) = 220$ $a^2 + b^2 = 22$	***2 marks – one mark per correct answer.***

Question 8 *(continued)*

	$(a+b)^2 = a^2 + 2ab + b^2$ $ab = \dfrac{(a+b)^2 - (a^2+b^2)}{2}$ $ab = \dfrac{36-22}{2} = 7$	***2marks – correct answer fully demonstrated*** ***1 mark – correct approach and significant progress made towards reaching correct answer.***
	${}^5\mathbf{C}_3 a^3x^3 + {}^5\mathbf{C}_3 b^3x^3 = {}^5\mathbf{C}_3(a^3+b^3)x^3$ $Coefficent\ = {}^5\mathbf{C}_3(a^3+b^3) = 10(a^3+b^3) = 900$ $a^3+b^3 = (a+b)(a^2-ab+b^2)$ $= (a+b)(a^2+b^2-ab)$ $= 6\times(22-5) = 90$ *or* $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$ $a^3+b^3 = (a+b)^3 - 3ab(a+b)$ $a^3+b^3 = 216 - 3\times 7\times 6 = 90$ $\therefore\quad Coefficent\ of\ \ x = 40\times 90 = 3600$	***2 marks – correct answer fully demonstrated.*** ***1 mark – correct process with arithmetic error.***